AD-A035 903 CARNEGIE-MELLON UNIV PITTSBURGH PA DEPT OF COMPUTER --ETC F/G 12/1
OPTIMAL RADIUS OF CONVERGENCE OF INTERPOLATORY ITERATIONS FOR O--ETC(U)
NOV 76 J F TRAUB, H WOZNIAKOWSKI N00014-76-C-0370

UNCLASSIFIED NL

| OF |

AD
A035903

END

DATE
FILMED
3-77

OPTIMAL RADIUS OF CONVERGENCE OF INTERPOLATORY ITERATIONS FOR OPERATOR EQUATIONS

J. F. Traub
Department of Computer Science
Carnegie-Mellon University
Pittsburgh, Pennsylvania 15213

H. Woźniakowski
Department of Computer Science
Carnegie-Mellon University
Pittsburgh, Pennsylvania 15213
(Visiting from the University of Warsaw)

November 1976

This research was supported in part by the National Science Foundation under Grant MCS75-222-55 and the Office of Naval Research under Contract N00014-76-C-0370, NR 044-422.

ABSTRACT

The convergence of the class of direct interpolatory iterations I_n for a simple zero of a non-linear operator F in a Banach space of finite or infinite dimension is studied.

A general convergence result is established and used to show that if F is entire the "radius of convergence" goes to infinity with n while if F is analytic in a ball of radius R the radius of convergence increases to at least R/2 with n.

1. INTRODUCTION

We study the convergence of the class of direct interpolatory iterations I_n, where I_n is of order n and $n \geq 3$, for a simple zero of a nonlinear operator F in a Banach space of finite or infinite dimension. For $n = 2$, see Traub and Woźniakowski [77a].

We establish a basic convergence theorem for I_n and apply it to two classes of problems. If F is an entire function of growth order ρ, then the "radius of convergence" goes to infinity at least as fast as $n^{1/\rho}$. We show this result is sharp for all odd n and $\rho = 1$. If F is analytic in a disk of radius R, then the "radius of convergence" increases to at least R/2 with n. This result is also sharp for all odd n.

The calculation of the next iterate, $x_{i+1} = I_n(x_i)$, requires the solution of a polynomial operator equation of degree n-1. In Traub and Woźniakowski [77b] we consider one way in which this can be done.

2. INTERPOLATORY ITERATION I_n

We consider the solution of the non-linear equation

$$(2.1)\quad F(x) = 0$$

where $F: D \subset B_1 \to B_2$ and B_1, B_2 are real or complex Banach spaces of dimension N, $N = \dim(B_1) = \dim(B_2)$, $1 \le N \le +\infty$. We solve (2.1) by one-point stationary iterations without memory in the sense of Traub [64]. Let x_i be a sufficiently close approximation to a simple zero α, i.e., $F(\alpha) = 0$ and $F'(\alpha)^{-1}$ exists and is bounded. Suppose that the next approximation x_{i+1} depends on the "standard information" $\mathfrak{N}$ on F, i.e.,

$$(2.2)\quad x_{i+1} = \varphi(x_i, \mathfrak{N}(x_i;F))$$

where φ is an iteration operator and

$$(2.3)\quad \mathfrak{N}(x_i;F) = \{F(x), F'(x), \dots, F^{(n-1)}(x)\} \quad \text{for } n \ge 2.$$

In Traub and Woźniakowski [76b] we showed that standard information is optimal (in a sense made precise in that paper) among all linear information sets which use n evaluations. Any iteration based on (2.3) has order of convergence no greater than n. This was first proved by Traub [64] and Kung and Traub [74] for scalar problems and by Woźniakowski [74] for multivariate and abstract problems, see also Traub and Woźniakowski [76a] who use a non-asymptotic definition of order and derive strict lower and upper bounds on complexity. In this paper we study the convergence of a class of iterations which use standard information. The optimal order is achievable by an interpolatory iteration I_n defined as follows:

(i) construct a polynomial w_i of degree $\leq n-1$ which agrees with the standard information on F, i.e.,

$$w_i^{(j)}(x_i) = F^{(j)}(x_i), \quad j = 0,1,\ldots,n-1, \tag{2.4}$$

(ii) define the next approximation x_{i+1} as a zero of w_i, $w_i(x_{i+1}) = 0$, with a certain criterion of its choice.

We shall write $x_{i+1} = I_n(x_i;F)$. From (2.4) we get

$$w_i(x) = F(x_i) + F'(x_i)(x-x_i) + \ldots + \frac{1}{(n-1)!} F^{(n-1)}(x_i)(x-x_i)^{n-1}. \tag{2.5}$$

For $n = 2$ we obtain Newton iteration since the unique zero of w_i is given by $x_{i+1} = x_i - F'(x_i)^{-1}F(x_i)$. Throughout this paper we assume $n \geq 3$. For $n = 2$, see Traub and Woźniakowski [77a].

Remark 2.1

Inverse interpolation can also be used in which case (2.4) is replaced by

$$w_i^{(j)}(F(x_i)) = g^{(j)}(F(x_i)), \quad j = 0,1,\ldots,n-1,$$

where $g(x) = F^{-1}(x)$ is the inverse operator to F. Define $x_{i+1} = w_i(0)$. The problem of solving the "polynomial operator equation" (2.5) is then eliminated but the derivatives of the inverse function must be computed. See Traub [64, Appendix B and Section 11.2] for the case $N < \infty$ and Brent and Kung [76] for $N = 1$ and n large. ∎

We deal with the character of convergence of the interpolatory iteration I_n. Let α be a simple zero of F, $e_i = \|x_i-\alpha\|$ and $J = \{x: \|x-\alpha\| \leq \Gamma\}$. Define

$$(2.6)\quad A_j = A_j(\Gamma) = \sup_{x\in J}\left\|F'(\alpha)^{-1}\frac{F^{(j)}(x)}{j!}\right\|,\quad j = 2,3,\ldots$$

whenever $F^{(j)}(x)$ exists. Let q be any number such that $0 < q < 1$ and let

$$\delta_n = \begin{cases} \max(\frac{nq}{1+q}, 1) & \text{if } N = +\infty \\ 1 & \text{if } N < +\infty, \end{cases}$$

$$\delta = \begin{cases} 2 & \text{if } N = +\infty \\ 1 & \text{if } N < +\infty . \end{cases}$$

<u>Theorem 2.1</u>

If F is n-times differentiable in J, $n \geq 3$, and

$$(2.7)\quad \frac{A_n(1+q)^n\Gamma^{n-1}}{q}\,\delta_n + A_2 q\Gamma\delta < 1,$$

$$(2.8)\quad x_0 \in J$$

then the polynomial w_i has a zero in $J_i = \{x: \|x-\alpha\| \leq q^i\Gamma\}$ for any i. Define x_{i+1} as any zero of w_i in J_i.

Then

$$(2.9)\quad \lim_i x_i = \alpha,\ e_{i+1} \leq \bar{q}_i e_i \quad \text{where}$$

$$\bar{q}_i = \frac{A_n(1+q)^{n-1}\Gamma^{n-1}q^{i(n-1)}}{1-A_n(1+q)^{n-1}\Gamma^{n-1}q^{i(n-1)}-A_2 q\Gamma q^i} \leq q,\quad \forall_i,$$

$$(2.10)\quad e_{i+1} \leq C_{i,n}e_i^n \quad \text{where}$$

$$C_{i,n} = A_n\left(1+\frac{e_{i+1}}{e_i}\right)^n/(1-A_2e_{i+1}),\quad \lim_i C_{i,n} = A_n,$$

$$(2.11)\quad x_{i+1}-\alpha = (-1)^n \frac{F'(\alpha)^{-1}}{n!} F^{(n)}(\alpha)(x_i-\alpha)^n + o(\|x_i-\alpha\|^n).$$

Proof

Define

$$(2.12)\quad R_j(x;y) = \int_0^1 F^{(j)}(y+t(x-y))(x-y)^j \frac{(1-t)^{j-1}}{(j-1)!}\,dt$$

for $x,y \in J$ and $j = 2$ and n. Assume by induction that $x_i \in J_i$. This holds for $i = 0$ since $J_0 = J$. From (2.4) we get the error formula

$$(2.13)\quad F(x) = w_i(x) + R_n(x;x_i),$$

see Rall [69, p.124]. We want to show that w_i has a zero in J_{i+1}. The equation $w_i(x) = 0$ in J_{i+1} is equivalent to $F(x) = R_n(x;x_i)$. Since $F(x) = F'(\alpha)(x-\alpha) + R_2(x;\alpha)$ we get the equation

$$(2.14)\quad x = H(x) \overset{df}{=} \alpha + F'(\alpha)^{-1}\{R_n(x;x_i) - R_2(x;\alpha)\}.$$

We verify $H(J_{i+1}) \subset J_{i+1}$. Let $x \in J_{i+1}$. Then from (2.6) and (2.12) we get

$$\|H(x)-\alpha\| \le A_n\|x-x_i\|^n + A_2\|x-\alpha\|^2 \le A_n(q^{i+1}+q^i)^n\Gamma^n + A_2 q^{2(i+1)}\Gamma^2 =$$

$$= q^{i+1}\Gamma\left(\frac{A_n(1+q)^n\Gamma^{n-1}}{q} q^{i(n-1)} + A_2 q\Gamma q^i\right) \le q^{i+1}\,\Gamma$$

due to (2.7). Thus $H(J_{i+1}) \subset J_{i+1}$.

If $N < +\infty$ then from the Brouwer fixed point theorem (see e.g., Ortega and Rheinboldt [70, p.161]) there follows the existence of the solution of (2.14), $x_{i+1} \in J_{i+1}$. If $N = +\infty$ we shall use the contraction mapping theorem (see e.g., Ortegaand Rheinboldt [70, p.120]). From (2.13) and (2.14) we get for $x \in J_{i+1}$

$$H'(x) = F'(\alpha)^{-1}\{(F'(x)-w_i'(x)) - (F'(x)-F'(\alpha))\}.$$

From (2.7) we get

$$\|H'(x)\| \le nA_n\|x-x_i\|^{n-1} + 2A_2\|x-\alpha\| \le nA_n(1+q)^{n-1}\,\Gamma^{n-1} + 2A_2q\Gamma < 1.$$

Hence H is contractive on J_{i+1} and there exists a unique solution x_{i+1} of (2.14) in J_{i+1} for any i.

Now let $N \le +\infty$. Setting $x = x_{i+1}$ in (2.14) we get

$$e_{i+1} \le A_n\|x_{i+1}-x_i\|^n + A_2e_{i+1}^2 \le A_n(1+q)^{n-1}\Gamma^{n-1}q^{i(n-1)}(e_{i+1}+e_i) + A_2q^{i+1}\Gamma e_{i+1}.$$

Thus $e_{i+1} \le \bar{q}_i e_i$ where $\bar{q}_i$ is given in (2.9) and $\bar{q}_i \le q$ due to (2.7). Hence $\lim_i e_i = 0$. Returning to (2.14) we have

$$e_{i+1} \le C_{i,n}e_i^n \text{ where } C_{i,n} = A_n(1+\frac{e_{i+1}}{e_i})^n/(1-A_2e_{i+1}).$$

Since $C_{i,n}$ is bounded, $e_{i+1}/e_i \le C_{i,n}e_i^{n-1} \to 0$. Hence $\lim_i C_{i,n} = A_n$ which proves (2.10). Next observe that

$$R_j(x_{i+1};x_i) = \frac{(-1)^j}{j!}F^{(j)}(\alpha)(x_i-\alpha)^j + o(\|x_i-\alpha\|^j).$$

Thus (2.14) for $x = x_{i+1}$ gives

$$x_{i+1}-\alpha = (-1)^n F'(\alpha)^{-1}\frac{F^{(n)}(\alpha)}{n!}(x_i-\alpha)^n + o(\|x_{i+1}-\alpha\| + \|x_i-\alpha\|^n)$$

which proves (2.11) and theorem 2.1. ■

Remark 2.2

For $N = +\infty$ we proved that there exists a unique zero of w_i in J_i for any i. For the multivariate case, $N < +\infty$, this need not be true. The next approximation x_{i+1} can be defined as any zero of w_i in J_i. However, it is easy to show that for large i the polynomial w_i has a unique zero in J_i. ■

Remark 2.3

The computation of x_{i+1} requires the solution of the "polynomial operator equation" $w_i(x) = 0$. There are a number of ways for dealing with the problem of solving this equation numerically. One is to apply a number of Newton iterations (say k) starting with x_i and taking the kth Newton iterate as x_{i+1}. Traub and Woźniakowski [77b] study the convergence and complexity of such "Interpolatory-Newton" iterations. ■

Remark 2.4

If we additionally assume that F is (n+1)-times differentiable in J then using the same technique it may be shown that

$$x_{i+1}-\alpha = (-1)^n F'(\alpha)\frac{1}{n!} F^{(n)}(\alpha)(x_i-\alpha)^n + \delta_{i,n}$$

where $\|\delta_{i,n}\| \le C\|x_i-\alpha\|^{n+1}$ and $C = C(A_2, A_n, A_{n+1})$. ■

In theorem 2.1 we assume that $x_0 \in J$. Note that the radius of J depends on F and q. For a given problem F one can ask what is the optimal value of q which maximizes Γ. In the next section we deal with that problem for any value of n.

3. BALL OF CONVERGENCE OF I_n

In the previous section we proved that if F is sufficiently regular and if an initial approximation x_0 belongs to the ball J then the interpolatory iteration I_n converges. Of course we would like to have J as large as possible. To make this idea more precise we define a ball of convergence as follows.

<u>Definition 3.1</u>

A ball $J_n = J_n(F) = \{x: \|x-\alpha\| < \Gamma_n^*\}$ where $\Gamma_n^* = \Gamma_n^*(F)$ is said to be <u>a ball of convergence</u> of the interpolatory iteration I_n for a function F if for any $x_0 \in J_n$ the sequence $x_{i+1} = I_n(x_i;F)$ converges to the solution α and for every $\epsilon > 0$, there exists x_0 such that $\|x_0-\alpha\| \le \Gamma_n^* + \epsilon$ and $x_{i+1} = I_n(x_i;F)$ is not convergent to α. Γ_n^* is called <u>the radius of the ball of convergence</u> or the radius of convergence for short. ■

Of course the ball of convergence can be defined for any iteration Φ in the same way. In fact it is sometimes more convenient to have the concept of <u>domain of convergence</u> D where $\alpha \in D$ and starting from any point $x_0 \in D$ the sequence $x_{i+1} = \Phi(x_i;F)$ converges to α. In general it is very hard to find D and therefore we restrict ourselves only to the case when D is a ball with center at α; compare Ortega and Rheinboldt [70, p.236].

Note that if F is a polynomial of degree $\le n-1$ then the interpolatory polynomial $w_i \equiv F$ and we get convergence for any x_0. This means $\Gamma_n^* = +\infty$ and $J_n(F) = B_1$. To exclude this exceptional case we shall assume throughout this section that F is not a polynomial.

Recall that $F: D \to B_2$, $D \subset B_1$ and

$$(3.1)\quad A_n = A_n(r) = \sup_{x\in J(r)} \left\| F'(\alpha)^{-1} \frac{F^{(n)}(x)}{n!} \right\|, \quad n = 2,3,\ldots$$

for r such that $J(r) = \{x\colon \|x-\alpha\| \le r\} \subset D$.

Define a function $g(r) = g(r,q)$ as

$$(3.2)\quad g(r) = \frac{A_n(r)(1+q)^n r^{n-1}}{q}\,\delta_n + A_2(r)qr\delta$$

where δ_n and δ are defined in Section 2. Note that g is strictly increasing and $g(0) = 0$, $g(+\infty) = +\infty$. Thus there exists a unique $r^* = r^*(q) > 0$ such that $g(r^*) = 1$. Let

$$(3.3)\quad r_n^* = \max_{0<q<1} r^*(q).$$

Applying theorem 2.1 with the q which maximizes r^* we can define the radius Γ in (2.7) as $\Gamma = r_n^* - \epsilon$ for any sufficiently small positive ϵ. From theorem 2.1 the radius Γ_n^* of the ball of convergence $J_n(F)$ can be bounded below by

$$(3.4)\quad \Gamma_n^* \ge r_n^*.$$

We shall see that for some problems $\Gamma_n^* \cong r_n^*$ for large n which indicates that theorem 2.1 is asymptotically sharp. Furthermore we shall show that for some problems r_n^* increases with n or even tends to infinity with n.

We consider two cases depending on the domain of F as follows:

(i) F is an entire function, i.e., $D = B_1$ and B_1, B_2 are Banach spaces over the complex or real field.

(ii) F is an analytic function in a finite domain $D = \{x\colon \|x-\alpha\| < R\}$ where $0 < R < +\infty$.

<u>Case (i)</u>. Let F be an entire function. Thus

$$(3.5)\quad F(x) = \sum_{i=1}^{\infty} \frac{1}{i!} F^{(i)}(\alpha)(x-\alpha)^i, \quad \forall x \in B_1.$$

<u>Definition 3.2</u>

We say F has the <u>growth order ρ</u>, $0 < \rho < +\infty$ and the <u>type τ, $\tau > 0$, of its order</u> if

$$\frac{\|F^{(i)}(\alpha)\|}{i!} \le M\left(\frac{\tau^i}{i!}\right)^{\frac{1}{\rho}}$$

for a constant M and $i = 0,1,\ldots$. ∎

Compare with the definition for the scalar case in Hille [62]. For the sake of simplicity we do not consider the growth order $\rho = 0$ or $\rho = +\infty$. However, it is possible to analyze such cases as well.

Define

$$(3.6)\quad f(z) = M\sum_{i=1}^{\infty}\left(\frac{\tau^i}{i!}\right)^{\frac{1}{\rho}} z^i, \quad z \in \mathbb{C}.$$

Then

$$(3.7)\quad \|F^{(j)}(x)\| \le f^{(j)}(\|x-\alpha\|)$$

for $x \in B_1$ and $j = 0,1,\ldots$.

We need bounds on the growth of $f^{(j)}$. From Hille [62, p.183] follows that there exists a constant $c_1 > 0$ such that

$$(3.8)\quad \max_{|z|\le r} |f(z)| = f(r) \le c_1 r^{\rho/2} \exp\left(\frac{\tau}{\rho} r^{\rho}\right), \quad \forall r > 0.$$

Lemma 3.1

Let $\mu = \max(0, 1-1/\rho)$. Then

$$(3.9)\quad \frac{f^{(n)}(r)}{n!} \le \frac{(2^{\mu}\tau^{\frac{1}{\rho}})^{n}}{(n!)^{1/\rho}} f(2^{\mu}r).$$

Proof

Note that

$$f^{(n)}(r) = MC^{n} \sum_{i=0}^{\infty} c_{i,n} \frac{(Cr)^{i}}{(i!)^{\nu}}$$

where $\nu = 1/\rho$, $C = \tau^{\nu}$ and $c_{i,n} = \left[\binom{n+i}{n} n!\right]^{1-\nu}$.

It may be shown that

$$n! \le \binom{n+i}{n} n! \le 2^{n+i}\, n!,\quad \forall n,i.$$

If $\rho < 1$ then $c_{i,n} \le (n!)^{1-\nu}$ and

$$\frac{f^{(n)}(r)}{n!} \le \frac{C^{n}}{(n!)^{1/\rho}} f(r)$$

which proves (3.9) with $\mu = 0$. Assume $\rho \ge 1$. Then $\mu = 1-\nu > 0$ and $c_{i,n} \le [n!\, 2^{n+i}]^{\mu}$. Hence

$$\frac{f^{(n)}(r)}{n!} \le \frac{(2^{\mu}C)^{n}}{(n!)^{1/\rho}} f(2^{\mu}r)$$

which proves (3.9).

We are ready to prove the follow[illegible] theorem.

Theorem 3.1

If F has the growth order ρ and the type τ of its order and if

$$(3.10)\quad q = q_n = \{(c_2 n)^{\frac{1+\rho/2}{\rho}} \exp(2^{\mu\rho}\tau c_2 n/\rho)\}^{-1} c_3$$

where c_2 and c_3 are any positive numbers such that

$$(3.11)\quad c_2 2^{\mu\rho}\tau \exp(2c_2 2^{\mu\rho}\tau+1) < 1,\ c_3 < 2^{\frac{1}{\rho}-2\mu}/(c_1\delta\tau^{2/\rho})$$

and μ, δ and c_1 are defined as above then theorem 2.1 holds with

$$(3.12)\quad \Gamma = \Gamma_n = (c_2 n)^{1/\rho}(1+o(1)),\ \forall n. \qquad \blacksquare$$

Proof

We want to estimate g defined by (3.2). Since $q_n = o(n)$, $\delta_n = 1$ in (2.7) and (3.2) for large n. After some algebraic manipulations we get from Lemma 3.2

$$g((c_2 n)^{\frac{1}{\rho}}) = 0\left(\{c_2 2^{\mu\rho}\tau \exp(2c_2 2^{\mu\rho}\tau+1)\}^{\frac{n}{\rho}} \frac{n^{\frac{n}{\rho}+1}}{(n!e^n)^{1/\rho}}\right) + \frac{(2^{\mu}\tau^{1/\rho})^2}{2^{1/\rho}}\,\delta c_1 c_3.$$

By Stirling's formula

$$n!e^n = n^{n+1/2}\sqrt{2\pi}\,(1+o(1))$$

and due to (3.11), $g((c_2 n)^{1/\rho}) < 1$ for large n. This means that (2.7) holds for $\Gamma = \Gamma_n = (c_2 n)^{1/\rho}(1+o(1))$ for every n. $\blacksquare$

Theorem 3.1 states a "type of global convergence" of the interpolatory iteration I_n. The iteration I_n is convergent for $\Gamma_n \cong (c_2 n)^{1/\rho}$ which tends to infinity with n and the growth of the radius depends on the growth order ρ.

From (3.3) and (3.4) we get

Corollary 3.1

If F has the growth order ρ then the radius Γ_n^* of the ball of convergence of the interpolatory iteration I_n satisfies

(3.13) $\Gamma_n^* \geq (c_2 n)^{1/\rho}(1+o(1)), \quad \forall n.$

We want to prove that (3.13) is sharp for n odd and $\rho = 1$.

Theorem 3.2

There exists a problem F of growth order $\rho = 1$ for which (3.13) is sharp for all odd n.

Proof

Let

(3.14) $F(x) = e^x - a, \quad a > 0,$

for real x. The growth order ρ and the type τ are now equal to unity. From Corollary (3.1) we get

$$\Gamma_n^*(F) \geq c_2 n(1+o(1))$$

where $c_2 \exp(2c_2+1) < 1$ which means $c_2 < 0.23$. We shall show that

(3.15) $\Gamma_n^*(F) \leq c_4 n(1+o(1))$

for n odd where $c_4 \exp(1+c_4) > 1$ which means $c_4 > 0.28$.

Let w be the interpolatory polynomial of degree $\leq n-1$ such that $w^{(j)}(x_0) = F^{(j)}(x_0)$ for $j = 0,1,\ldots,n-1$. Then

$$w(x) = e^{x_0} S_{n-1}(x-x_0) - a$$

where

$$S_k(x) \overset{\text{df}}{=} \sum_{i=0}^{k} \frac{x^i}{i!} = \int_0^\infty \frac{(x+t)^k}{k!} e^{-t} dt,$$

see Newman and Rivlin [72]. From this it follows that S_{2k} does not have real zeros and S_{2k-1} does have a unique real zero which we label by z_{2k-1}. It is known that

$$z_{2k-1} = -c_5(2k-1)(1+o(1)), \quad \forall k,$$

where $c_5 \exp(1+c_5) = 1$, $c_5 \doteq 0.28$, see Szegö [24] and Rosenblum [57]. Observe that $S'_{2k}(x) = S_{2k-1}(x)$ which gives

$$\min_x S_{2k}(x) = S_{2k}(z_{2k-1}) = \frac{z_{2k-1}^{2k}}{(2k)!}.$$

We want to find x_0 such that the polynomial w does not have real zeros which means that the interpolatory iteration is not defined at x_0 and $x_0 \notin J_n(F)$. The equation $w(x) = 0$ is equivalent to

$$S_{n-1}(x-x_0) = ae^{-x_0}.$$

Let $x_0 = c_4 n$. Then

$$(3.16) \quad S_{n-1}(x-x_0) - ae^{-x_0} \geq \frac{z_{n-2}^{n-1}}{(n-1)!} - ae^{-c_4 n}.$$

Note that $\frac{|z_{n-2}|}{\sqrt[n-1]{(n-1)!}} = c_5 e(1+o(1))$ and $\sqrt[n-1]{ae^{-c_4 n}} = e^{-c_4}(1+o(1))$. Since $c_5 e = \exp(-c_5)$ and $c_5 < c_4$ then (3.16) is always positive for large n. Hence (3.15) and theorem 3.2 are proven. ■

However, it may be shown that $\Gamma_n^*(F) = +\infty$ for n even for the problem (3.14). The sharpness of (3.13) for n even or $\rho \neq 1$ is open.

<u>Case (ii)</u>. Let F be analytic in D where

$$D = \{x: \|x-\alpha\| < R\}, \quad 0 < R < +\infty.$$

This means that for any sufficiently small $\epsilon > 0$ there exists $M = M(F;\epsilon)$ such that

$$(3.17) \quad \frac{\|F^{(i)}(\alpha)\|}{i!} \le M\,C^{i} \quad \text{where} \quad C = \frac{1}{R-\epsilon}, \quad i = 2,3,\ldots$$

<u>Theorem 3.3</u>

If F satisfies (3.17) and if

$$(3.18) \quad q = q_n = \frac{1}{MCn}$$

then theorem 2.1 holds with

$$(3.19) \quad \Gamma = \Gamma_n = \frac{1-\epsilon}{2}(R-\epsilon)(1+o(1)).$$

<u>Proof</u>

Define

$$f(z) = M\sum_{i=0}^{\infty}(Cz)^{i} = \frac{M}{1-Cz} \quad \text{for } |z| \le \frac{1}{C} = R-\epsilon.$$

From (3.17) we get

$$(3.20) \quad \|F^{(i)}(x)\| \le f^{(i)}(\|x-\alpha\|) = \frac{i!\,MC^{i}}{(1-C\|x-\alpha\|)^{i+1}}$$

for any x such that $\|x-\alpha\| < R-\epsilon$ and $i = 2,3,\ldots$.

Let $r = r_n = \frac{1-\epsilon}{2C} = \frac{1-\epsilon}{2}(R-\epsilon)$. From (3.2) and (3.20) we have

$$g(r) = O\left(n\left(\frac{1-\epsilon}{1+\epsilon}\right)^{n} + \frac{1}{n}\right) < 1$$

for large n. This means that (2.7) holds for $\Gamma = \Gamma_n = \frac{1-\epsilon}{2}(R-\epsilon)(1+o(1))$ for all n. ■

<u>Remark 3.1</u>

It is possible to get a slightly sharper estimate of Γ_n in (3.13). It may be shown that

$$\Gamma_n = \frac{R-\epsilon}{2}\left(1 - \frac{\ln(nc_6)}{2n}(1+o(1))\right)$$

where $c_6 = c_6(M,C)$ is a positive constant. ■

Since ϵ can be arbitrarily small, theorem 3.3 states that $\Gamma_n \cong \frac{1}{2}R$ for large n. This means that the radius Γ_n is about one half of the domain radius R. Once more this gives a "type of global convergence". From (3.4) we get

<u>Corollary 3.2</u>

If F satisfies (3.17) then the radius Γ_n^* of the ball of convergence of the interpolatory iteration I_n satisfies

$$(3.21) \quad \Gamma_n^* \geq \frac{1}{2}R(1+o(1)), \quad \forall n. \quad ■$$

We now show that, in general, (3.21) is sharp for n odd.

<u>Theorem 3.4</u>

There exists a problem F for which (3.21) is sharp for all odd n.

<u>Proof</u>

Let

$$(3.22) \quad F(x) = \frac{1}{1-Cx} - 1, \qquad 0 < C < 1$$

for real x such that $|x| < R = \frac{1}{C}$. We shall show that

$$\Gamma_n^*(F) \le \frac{1}{2}R, \quad \forall\ n \text{ odd.}$$

The interpolatory polynomial w is given by

$$w(x) = \sum_{i=0}^{n-1} \frac{C^i}{(1-Cx_0)^{i+1}} (x-x_0)^i - 1, \quad n \text{ odd.}$$

The equation $w(x) = 0$ is equivalent to

$$(3.23) \quad \left[\frac{C(x-x_0)}{1-Cx_0}\right]^n = Cx.$$

Let $x_0 = \frac{1}{2C} = \frac{1}{2}R$. Then (3.23) becomes

$$(3.24) \quad (2Cx-1)^n = Cx.$$

It is straightforward to verify that (3.24) does not have zeros in [-R,R]. This implies $x_0 \notin J_n(F)$ and

$$\Gamma_n^*(F) \le \frac{1}{2}R, \quad \forall\ n \text{ odd.}$$

However, it may be shown that $\Gamma_n^*(F) = R$ for n even. Thus the sharpness of (3.21) is open for n even.

ACKNOWLEDGMENT

We would like to thank H. T. Kung for his comments on this paper.

REFERENCES

Brent and Kung [76] — Brent, R. P. and Kung, H. T., "Fast Algorithms for Manipulating Formal Power Series," Dept. of Computer Science Report, Carnegie-Mellon University, 1976.

Hille [62] — Hille, E., Analytic Function Theory, Vol. II, Ginn, Boston, 1962.

Kung and Traub [74] — Kung, H. T. and Traub, J. F., "Optimal Order of One-Point and Multipoint Iteration," Journal of ACM, 21, 1974, 643-651.

Newman and Rivlin [72] — Newman, D. J. and Rivlin, T. J., "The Zeros of the Partial Sums of the Exponential Function," J. App. Theory, 5, (1972), 405-412.

Ortega and Rheinboldt [70] — Ortega, J. M. and Rheinboldt, W. C., Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York, 1970.

Rall [69] — Rall, L. B., Computational Solution of Nonlinear Operator Equations, John Wiley and Sons, New York, 1969.

Rosenbloom [55] — Rosenbloom, P. C., "Distribution of Zeros of Polynomials," in Lectures on Functions of a Complex Variable, edited by W. Kaplan, Ann Arbor, 1955, 265-285.

Szegö [24] — Szegö, G., "Über eine Eigenschaft der Exponentialreihe," Berlin Math. Ges. Sitzungsber, 23 (1924), 50-64.

Traub [64] — Traub, J. F., Iterative Methods for the Solution fo Equations, Prentice Hall, 1964.

Traub and Woźniakowski [76a] — Traub, J. F. and Woźniakowski, H., "Strict Lower and Upper Bounds on Iterative Computational Complexity," in Analytic Computation Complexity, edited by J. F. Traub, Academic Press, 1976, 15-34.

Traub and Woźniakowski [76b] — Traub, J. F. and Woźniakowski, H., "Optimal Linear Information for the Solution of Nonlinear Equations," in Algorithms and Complexity: New Directions and Recent Results, edited by J. F. Traub, Academic Press, 1976.

Traub and Woźniakowski [77a]	Traub, J. F. and Woźniakowski, H., "Convergence and Complexity of Newton Iteration for Operator Equations," Dept. of Computer Science Report, Carnegie-Mellon University, 1977.
Traub and Woźniakowski [77b]	Traub, J. F. and Woźniakowski, H., "Convergence and Complexity of Interpolatory-Newton Iterations in a Banach Space," Dept. of Computer Science Report, Carnegie-Mellon University, 1977.
Woźniakowski [74]	Woźniakowski, H., "Maximal Stationary Iterative Methods for the Solution of Operator Equations," SIAM J. Numer. Anal., 11, 5, (October, 1974) 934 - 949.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER 9
4. TITLE (and Subtitle) 6 OPTIMAL RADIUS OF CONVERGENCE OF INTERPOLATORY ITERATIONS FOR OPERATOR EQUATIONS.		5. TYPE OF REPORT & PERIOD COVERED Interim rept.,
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) 10 J. F Traub ~~and~~ H Wozniakowski		8. CONTRACT OR GRANT NUMBER(s) 15 N00014-76-C-0370, ~~NR 044-422~~
9. PERFORMING ORGANIZATION NAME AND ADDRESS Carnegie-Mellon University Computer Science Dept. Pittsburgh, PA 15213		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Arlington, VA 22217		12. REPORT DATE 11 November 1976
		13. NUMBER OF PAGES 22
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) 12 23p.		15. SECURITY CLASS (of this report)
		15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

I sub n

20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The convergence of the class of direct interpolatory iterations I_n for a simple zero of a non-linear operator F in a Banach space of finite or infinite dimension is studied. A general convergence result is established and used to show that if F is entire the "radius of convergence" goes to infinity with n while if F is analytic in a ball of radius of convergence increases to at least R/2 with n.

)a

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

403081

VB